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THE MOTION OF A POINT MASS IN A MEDIUM WITH A SQUARE LAW OF DRAG†

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A local analytical solution of the problem of the motion of a point mass in a medium with a square law of drag is constructed, and simple analytical formulae are obtained for the main parameters of the trajectory of the point mass. It is shown that the trajectory can be calculated with high accuracy using a simple and economical algorithm. Examples are given. © 2001 Elsevier Science Ltd. All rights reserved.

The problem of the motion of a point mass in a medium with drag (the fundamental problem of external ballistics) has been considered in many papers [1–9], beginning with Euler's [1], which developed a method of investigation, that later became known as Euler's method. This method was used by him to solve the problem of the exact integration of the equation of the velocity hodograph for a quadratic relation between the drag of the medium and the velocity of the point mass. Later the problem was solved repeatedly by different methods (the small-parameter method, the constant-variation method, the perturbation method and Chaplygin's method), but an accurate analytical solution was not found. Various forms of approximate solutions have been obtained [2–8], but they are not very convenient to use, since they are described by extremely lengthy formulae; moreover, they are often obtained assuming that the drag is small compared with the gravity force [4, 7]. However, this assumption is not satisfied for the motion of certain objects, for example, a baseball or a golf ball, and hence various numerical schemes have been developed to integrate the equations of motion. Nevertheless, Euler's approach enables simple analytical expressions to be obtained for the time and coordinates of the point mass in the form of a function of the known velocity of the point mass, by means of which one can construct the trajectory of the point mass and obtain the fundamental parameters of motion with high accuracy. The analytical solution proposed below differs from the other solutions in the simplicity of the formulae, ease of use, high accuracy and the absence of constraints imposed on the value of the drag.

1. EQUATIONS OF MOTION AND CONSTRUCTION OF THE TRAJECTORY

The problem of the motion of a point mass in a medium with a square law of the drag $R = mgkV^2$, with the usual assumptions, reduces to solving the well-known system of differential equations [2]

$$\dot{V} = -g\sin\theta - gkV^2,\quad \dot{\theta} = -g\cos\theta / V \qquad (1.1)$$
$$\dot{x} = V\cos\theta,\quad y = V\sin\theta$$

Here V is the velocity of the point mass, m is its mass, θ is the slope of the trajectory to the horizontal, g is the acceleration due to gravity, x, y are the Cartesian coordinates of the point, and k is a coefficient of proportionality (Fig. 1).

The well-known solution [2] of Eqs (1.1), obtained by Euler's method, consists of an explicit analytical dependence of the velocity on the slope of the trajectory and three quadratures

$$V(\theta) = \frac{V_0\cos\theta_0}{\cos\theta\sqrt{1 + kV_0^2\cos^2\theta_0(f(\theta_0) - f(\theta))}} \qquad (1.2)$$
$$f(\theta) = \frac{\sin\theta}{\cos^2\theta} + \ln\operatorname{tg}\left(\frac{\theta}{2} + \frac{\pi}{4}\right)$$

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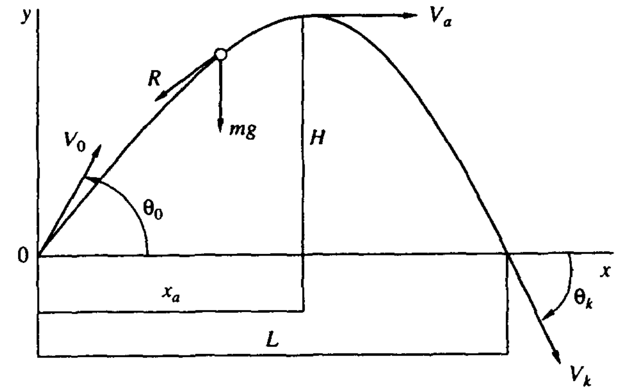

Fig. 1

$$t = t_0 - \frac{1}{g}\int_{\theta_0}^{\theta} \frac{V}{\cos\theta} d\theta, \quad x = x_0 - \frac{1}{g}\int_{\theta_0}^{\theta} V^2 d\theta \tag{1.3}$$

$$y = y_0 - \frac{1}{g}\int_{\theta_0}^{\theta} V^2 \operatorname{tg}\theta d\theta$$

Here V_0 and θ_0 are the initial values of the velocity and the slope of the trajectory respectively, t_0 is the initial value of the time, and x_0 and y_0 are the initial values of the coordinates of the point mass (in general non-zero).

The integrals on the right-hand sides of (1.3) are not taken in finite form. Hence, to determine the variables t, x and y we must either integrate (1.1) numerically or evaluate the definite integrals (1.3).

It turns out [9] that, using a special form of organised integration of quadratures (1.3) by parts in a fairly small interval $[\theta_0,\theta]$, the variables t, x and y can be written in the form

$$t = t_0 + \frac{2(V_0 \sin\theta_0 - V\sin\theta)}{g(2+\varepsilon)} \tag{1.4}$$

$$x = x_0 + \frac{V_0^2 \sin 2\theta_0 - V^2 \sin 2\theta}{2g(1+\varepsilon)}, \quad y = y_0 + \frac{V_0^2 \sin^2\theta_0 - V^2 \sin^2\theta}{g(2+\varepsilon)}$$

$$\varepsilon = k(V_0^2 \sin\theta_0 + V^2 \sin\theta) \tag{1.5}$$

We will obtain the first of formulae (1.4). The method of calculating the quadratures is based on the use of the relation between an auxiliary variable $u = V\cos\theta$ and the independent variable θ. This relation has the following differential form [2]

$$\frac{du}{u^3} = k\frac{d\theta}{\cos^3\theta} \tag{1.6}$$

We will consider the first of quadratures (1.3) and we write it, using the relation $u = V\cos\theta$, in the form

$$t = t_0 - \frac{1}{g}\int_{\theta_0}^{\theta} \frac{V}{\cos\theta} d\theta = t_0 - \frac{1}{g}\int_{\theta_0}^{\theta} u\frac{d\theta}{\cos^2\theta} \tag{1.7}$$

We take the integral (1.7) by parts

$$t = t_0 - \frac{u\operatorname{tg}\theta}{g}\Bigg|_{\theta_0}^{\theta} + \frac{1}{g}\int_{\theta_0}^{\theta} \operatorname{tg}\theta du = t_0 - \frac{V\sin\theta}{g}\Bigg|_{\theta_0}^{\theta} + \frac{1}{g}\int_{\theta_0}^{\theta} \operatorname{tg}\theta du$$

Using relation (1.6) we convert the last term

$$\frac{1}{g}\int_{\theta_0}^{\theta} \mathrm{tg}\,\theta du = \frac{k}{g}\int_{\theta_0}^{\theta} V^3 \mathrm{tg}\,\theta d\theta = -k\int_{\theta_0}^{\theta} V^2 \sin\theta dt$$

Hence

$$t = t_0 - \frac{V\sin\theta}{g}\Big|_{\theta_0}^{\theta} - ktV^2\sin\theta\Big|_{\theta_0}^{\theta} + k\int_{\theta_0}^{\theta} td(V^2\sin\theta) \tag{1.8}$$

Suppose the range of integration $\theta - \theta_0 = \Delta\theta$ is fairly small. Then the integral in (1.8) can be calculated as the area of a trapezium with bases t_0 and t and height $h = V^2\sin\theta - V_0^2\sin\theta_0$. We have

$$k\int_{\theta_0}^{\theta} td(V^2\sin\theta) \approx \frac{k(t_0+t)}{2}\int_{\theta_0}^{\theta} d(V^2\sin\theta) = \frac{1}{2}k(t_0+t)(V^2\sin\theta - V_0^2\sin\theta_0)$$

As a result, formula (1.8) takes the form

$$t\left(1+\frac{\varepsilon}{2}\right) = t_0\left(1+\frac{\varepsilon}{2}\right) + \frac{V_0\sin\theta_0 - V\sin\theta}{g}$$

(the variable ε is defined by relation (1.5)). Finally

$$t = t_0 + \frac{2(V_0\sin\theta_0 - V\sin\theta)}{g(2+\varepsilon)}$$

We can similarly derive the other two formulae of (1.4).

Hence, in a small interval $[\theta_0,\theta]$ the trajectory of the point mass can be approximated by Eqs (1.4). These formulae have a local form. We can calculate the whole trajectory very accurately in steps by calculating $V(\theta)$, $t(\theta)$, $x(\theta)$, $y(\theta)$ using Eqs (1.2) and (1.4) at the right-hand end of the interval $[\theta_0,\theta]$ and taking them as the initial values for the following step

$$V_0 = V(\theta),\ \ t_0 = t(\theta),\ \ x_0 = x(\theta),\ \ y_0 = y(\theta)$$

This cyclical procedure replaces both numerical integration of system (1.1) and the evaluation of the integrals (1.3). The smaller the value of k the greater the range $[\theta_0,\theta]$of applicability of the formulae obtained. When $k = 0$, i.e. when there is no drag, formulae (1.4) reduce to the well-known accurate formulae of the theory of the parabolic motion of a point mass and become valid for any values of θ_0 and θ. Moreover, formulae (1.4) are accurate in those finite intervals of $[\theta_0,\theta]$ where the variables t, x and y depend linearly on the auxiliary variable
$z = V^2\sin\theta$.

As calculations show, the trajectory obtained by integrating system of equations (1.1) and the trajectory constructed using formulae (1.2) and (1.4), are identical. Here, to construct the trajectory it is sufficient to use a step $\Delta\theta = \theta - \theta_0$ of the order of 0.1°. Below we give an example of a calculation using formulae (1.2) and (1.4) of the trajectory of a baseball with the following initial values of V_0 and θ_0 and a value of the coefficient k [3]

$$V_0 = 44.69\ \text{m/s},\ \theta_0 = 60°,\ k = 0.000548\ \text{s}^2/\text{m}^2,\ g = 9.81\ \text{m/s}^2 \tag{1.9}$$

θ, deg	60	30	0	–30	–60	–70.11
V, m/s	44.69	18.36	14.71	15.88	23.50	29.35
t, s	0	2.153	3.052	3.889	5.391	6.533
x, m	0	39.77	53.49	65.40	84.62	97.04
y, m	0	48.48	52.55	49.18	27.57	3×10^{-4}

Note that in this example, at the initial stage of the motion of the ball, the drag is greater than the gravity force, and during the motion its value changes by an order of magnitude.

2. ANALYTICAL FORMULAE FOR DETERMINING THE MAIN PARAMETERS OF MOTION OF THE POINT MASS

Equations (1.4) enable us to obtain simple analytical formulae for the main parameters of motion of the point mass. In Fig. 2, for the values given in (1.9), we have drawn a graph of the coordinates y against the auxiliary dimensionless variable $R_y = -kV^2 \sin\theta$, where R_y is the projection of the normalized drag of the medium on the y axis. The variable R_y is similar to the above-mentioned variable z. It can be seen that, both at the ascending stage (the left part of the graph) and the descending stage (the right part) this graph is close to linear. Hence it follows that the maximum height of ascent of the point mass H can be obtained approximately using formula (1.4) for y in the finite interval $[\theta_0,\theta]$, i.e. by taking $\theta = 0$ in this formula.

From the relation for the maximum height of ascent H we can derive comparatively simple approximate analytical formulae for the other parameters of motion of the point mass. We will give a complete summary of the formulae for the maximum height of ascent of the point mass H, the velocity at the vertex of the trajectory V_a, the time of ascent t_a, the range L, the time of motion t, the abscissa of the vertex of the trajectory x_a, the final velocity V_k and the angle of incidence θ_k (see Fig. 1):

$$\begin{gathered}
H = \frac{V_0^2 \sin^2\theta_0}{g(2 + kV_0^2 \sin\theta_0)} \\
V_a = \frac{V_0 \cos\theta_0}{\sqrt{1 + kV_0^2(\sin\theta_0 + \cos^2\theta_0 \ln\operatorname{tg}(\theta_0/2 + \pi/4))}} \\
t_a = \sqrt{\frac{2H}{g} + \left(\frac{kHV_a}{2}\right)^2} - \frac{kHV_a}{2}, \quad L = 2V_a\sqrt{\frac{2H}{g}} \\
T = t_a + \frac{2H}{gt_a}, \quad x_a = \frac{L(T - t_a)}{\sqrt{2Tt_a}} \\
V_k = \frac{\sqrt{a^2(1-c^2) + d^2(1-b^2) + 2abcd} - (ab + cd)}{1 - b^2 - c^2} \\
\theta_k = \arcsin\left[c(1+b) - \frac{d}{V_k}\right] \\
\left(a = \frac{L - x_a}{T - t_a}, \quad b = \frac{kg(L - x_a)}{2}, \quad c = \frac{kgH}{2}, \quad d = \frac{gT}{2}\right)
\end{gathered} \tag{2.1}$$

When $k = 0$ formulae (2.1) reduce to the corresponding formulae of the theory of the parabolic motion of a point mass.

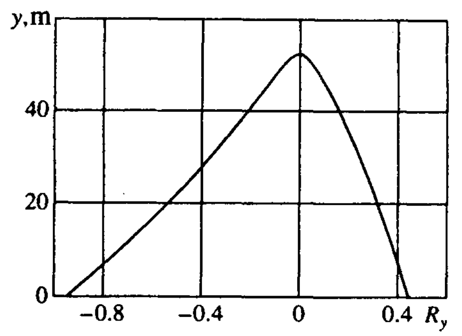

Fig. 2

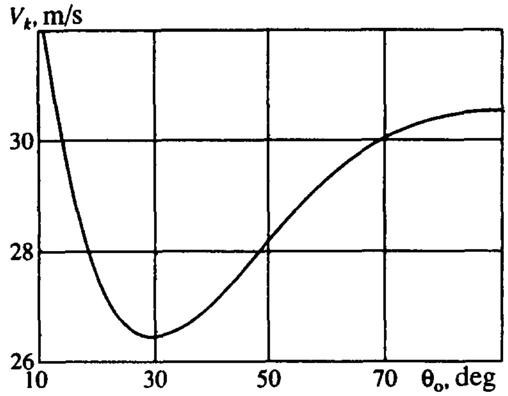

Fig. 3

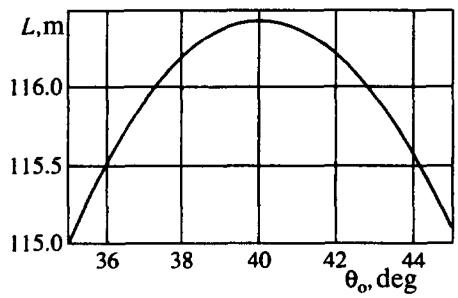

Fig. 4

3. RESULTS AND CONCLUSIONS

Formulae (2.1) enable us to calculate the fundamental parameters of motion of a point mass directly from the initial data V_0 and θ_0, as in the theory of parabolic motion. As an example of the use of formulae (2.1) we calculated the motion of a baseball with the following initial conditions

$$V_0 = 39.62 \text{ m/s}, \ \theta_0 = 60°$$

The following results were obtained (we indicate in parenthesis the value of the deviation from the exact value of the parameter, obtained using formulae (1.2) and (1.4) with a step of $\Delta\theta = 0.1°$)

$$H = 43.72 \text{ m} (-0.6\%), \qquad V_a = 13.91 \text{ m/s} (0.0\%)$$
$$t_a = 2.823 \text{ s} (0.0\%), \qquad L = 83.06 \text{ m} (-0.9\%)$$
$$T = 5.980 \text{ s} (0.0\%), \qquad x_a = 45.12 \text{ m} (-1.0\%)$$
$$V_k = 27.66 \text{ m/s} (-0.1\%), \qquad \theta_k = -68.61° (-0.3\%)$$

Analytical formulae (2.1) enable a parametric analysis and an optimization of the problem to be carried out. In Figs 3 and 4 we have drawn graphs of $V_k(\theta_0)$ and $L(\theta_0)$ where $V_0 = 44.69$ m/s. It can be seen that the final velocity is a minimum when $\theta_0 = 30°$ and the maximum range is obtained when $\theta_0 = 40°$, whereas when $k = 0$ the value of the maximum range $L_{\max} = L\ (\theta_0 = 45°)$.

Hence, the simple and convenient analytical formulae obtained by Euler's method enable a solution to be constructed which is no less accurate than the numerical solution of the equations of motion.

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